CHAPTER FOUR

APPLICATIONS TO APPROXIMATIONS

Now we have learned some basics about normed spaces and the inner product spaces. One thing important is that lots of popular spaces fall into the category of those above mentioned two spaces.

For example, $C[0,1]$ is a Banach space (when equipped with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$), and $C[0,1]$ is an inner product space, when equipped with inner product

$$\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt, \quad \forall f, g \in C[0,1].$$

One very important and useful task in those function spaces is ~~how~~ to approximate a function using "relatively simple" functions.

This idea is never new. For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

the series expansion of e^x, can be viewed as approximating e^x with polynomials.

In this chapter, we will talk about how to do such "approximations" in a general sense, a.k.a, in normed spaces or in inner product spaces.

Approximations in Hilbert Spaces

We start with the easy case, approximations in Hilbert spaces.

This case is easy, because of the following Bessel Inequality

$$\|x\|^2 \geq \sum_{i \in \mathcal{A}} |\langle x, e_i \rangle|^2$$

where $\langle e_i \rangle_{i \in \mathcal{A}}$ are mutually orthogonal and each e_i is a unit vector.

Besides, if $\langle e_i \rangle_{i \in \mathcal{A}}$ is further a basis (that is,

$$\{e_i : i \in \mathcal{A}\}^{\perp} = 0),$$

then

$$\|x\|^2 = \sum_{i \in \mathcal{A}} |\langle x, e_i \rangle|^2 .$$

Due to those good properties, given any $x \in H$, where H is a Hilbert space, and given an orthogonal basis $\{e_i : i \in \mathbb{N}\}$

[Note: For simplicity, we assume that H is separable, such that we can find a countable basis for H. This assumption is satisfied for most of the function spaces we are interested in for this class]

Under the above settings, we have

$$\sum_{i=1}^{n} \langle x, e_i \rangle e_i \xrightarrow{n \to \infty} x \qquad \forall x \in H.$$

In other words, we can approximate x with finite linear combinations of $\{e_i\}_{i \in \mathbb{N}}$. So the main thing is to find $\{e_i\}_{i \in \mathbb{N}}$ s.t

1) $\{e_i\}_{i \in \mathbb{N}}$ is a basis of H

2) Each e_i is "simple".

We will use two examples to demonstrate the idea.

Example 1. $H = L^2(\mathbb{T})$.

In this case,

$$H = \{ f : f \text{ is measurable on } \mathbb{T} \text{ and } \int |f \bar{f}| \, dt < \infty \}.$$

One orthogonal basis of this H is

$$\{ z^n : n \in \mathbb{Z} \}.$$

The orthogonality is easy to check. In fact, for any $m \neq n$,

$$\langle z^m, z^n \rangle = \frac{1}{2\pi}\int_{\mathbb{T}} z^m \cdot \overline{z^n}\, dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{T}} z^{m-n}\, dt$$

$$= 0.$$

When $m = n$, one can check that

$$\langle z^m, z^n \rangle = 1.$$

One thing remains to be checked is that

$$\{e_i\}_{i \in \mathbb{N}}^{\perp} = 0,$$

which follows from the (generalized version) of the Weierstrass Theorem.

<u>Observation</u>: Each e_i is "relatively simple", as each z^n is a complex polynomial.

<u>Conclusion</u>: $\forall\, f \in L^2(\mathbb{T})$, we have

$$f = \lim_{n\to\infty} \sum_{k=-n}^{n} \left[\frac{1}{2\pi}\int f(e^{it})\, e^{-ikt}\, dt\right] \cdot e^{ikt}.$$

Example 2 . $L^2[0,1]$.

$$L^2[0,1] \equiv \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a measurable real function on } [0,1] \\ \text{such that } \int_0^1 |f|^2 dt < \infty \end{array} \right\}.$$

According to Weistrass Thm. $P[0,1]$ is dense in $L^2[0,1]$.

where $P[0,1]$ is made up of all polynomials.

Easy to check that

$$\text{span}\langle 1, x, x^2, \ldots \rangle = P[0,1].$$

But $1, x, x^2, x^3, \ldots$ is not a basis b/c

$\langle x^m, x^n \rangle$ might not always be zero for $m \neq n$.

We will apply the classical Gram-Schmidt process to solve this issue.

Let $e_0 = 1$.

We can not set $e_1 = x$. b/c $\langle e_0, x \rangle \neq 0$.

Let $f_1 = x - \langle x, e_0 \rangle e_0$. We can check that

$$\langle f_1, e_0 \rangle = \langle x - \langle x, e_0 \rangle e_0, e_0 \rangle$$

$$= \langle x, e_0 \rangle - \langle x, e_0 \rangle \langle e_0, e_0 \rangle$$

$$= \langle x, e_0\rangle - \langle x, e_0\rangle$$
$$= 0.$$

Note that $\|f_1\|$ might not be 1. Let

$$e_1 = f_1 / \|f_1\|.$$

One can immediately check that

$$\langle e_0, e_1\rangle = 0 \text{ and } \|e_0\| = \|e_1\| = 1.$$

Now, let

$$f_2 = x^2 - \left[\langle x^2, e_0\rangle e_0 + \langle x^2, e_1\rangle e_1\right].$$

and let

$$e_2 = f_2 / \|f_2\|.$$

We have

$$\langle e_0, e_1\rangle = 0, \ \langle e_1, e_2\rangle = 0, \ \langle e_0, e_2\rangle = 0$$

$$\|e_0\| = \|e_1\| = \|e_2\| = 1.$$

and

$$\text{span}\langle e_0\rangle = \text{span}\langle 1\rangle$$

$$\text{span}\langle e_0, e_1\rangle = \text{span}\langle 1, x\rangle.$$

$$\text{span}\langle e_0, e_1, e_2\rangle = \text{span}\langle 1, x, x^2\rangle.$$

Keep applying this process, we get a basis $\{e_i : i \in \mathbb{N}\}$ of $L^2[0,1]$.

It then follows that

$$f = \lim_{n\to\infty} \sum_{i=0}^{n} \langle f, e_i \rangle e_i \qquad \forall f \in L^2[0,1]$$

Note that each e^k is a polynomial of degree k. We just successfully approximated f using polynomials in the norm $L^2[0,1]$.

Summary :

To approximate f in a Hilbert space H, we just need to find a "proper" basis for H.

Approximations on Banach Spaces

To do approximations on Banach spaces is not as simple as the Hilbert space case. That is because we do not have

$$\|x\|^2 = \sum_{i \in A} |\langle x, e_i \rangle|^2 \quad \text{w/ } \{e_i : i \in A\} \text{ being a basis}.$$

Thus we can NOT claim

$$x = \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle e_i .$$

In fact, we donot even have the "basis" in Banach spaces.

Example Consider $C(\mathbb{T})$ with norm given as

$$\|f\| = \sup_{z \in \mathbb{T}} |f(z)| \quad . \quad \forall f \in C(\mathbb{T}) .$$

It is a Banach space (but not a Hilbert space). If we mimick the $L^2(\mathbb{T})$ as before, and define

$$f_n = \sum_{k=-n}^{n} \langle f, z^k \rangle z^k .$$

We might not always have

$f_n \longrightarrow f$ in the above defined norm.

This fact is a well-known ~~fact~~ in Fourier Analysis. In fact, from certain point of view. Fourier analysis is just to deal with such issues.

Luckily for us (will be explained in detail), such approximations, after certain fix, are still possible.

For example. for any $f \in$ ~~$C[0,1]$~~ $C(\mathbb{T})$, define f_n by

$$f_n(t) = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \cdot \left[\frac{1}{2\pi} \int f(t) e^{-ijt} \, dt\right] \cdot e^{ijt}.$$

Then we have

$$f_n \longrightarrow f \text{ as } n \to \infty$$

~~in~~ under the norm $\|\cdot\|$ as defined above.

Note that

$$f_n = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \langle f, z^n \rangle \cdot z^n$$

It is not the same as

$$\sum_{-n}^{n} \langle f, z^n \rangle \cdot z^n.$$

The coefficients $\left(1 - \frac{|j|}{n+1}\right)$ [The Fejér kernel] make all the difference here.

More on Fourier Series (and a bit of wavelets)

Fourier Series on $\mathbb{T}$ (the unit circle).

Denote by $L^1(\mathbb{T})$ the space of all complex-valued L^1-integrable (w.r.t Lebesgue measure) functions on $\mathbb{T}$.

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|\,dt$$

<u>DEFN</u>. A trigonometric series on $\mathbb{T}$ is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}$$

For $f \in L^1(\mathbb{T})$, define $\hat{f}(n)$ [the n-th Fourier Coefficient] as

$$\hat{f}(n) = \frac{1}{2\pi}\int f(t)\, e^{-int}\,dt.$$

<u>DEFN</u>. The Fourier series $S[f]$ of a function $f \in L^1(\mathbb{T})$ is the trigonometric series

$$S[f] \sim \sum_{-\infty}^{\infty} \hat{f}(n)\, e^{int}$$

<u>Some basic facts</u>.

Thm. Let $f, g \in L^1(\mathbb{T})$, then

(a) $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$

(b) $\forall \alpha \in \mathbb{C}$

$$\widehat{(\alpha f)}(n) = \alpha \hat{f}(n)$$

(c) If $\bar{f}$ is the complex conjugacy of f, then

$$\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}.$$

(d) Let $f_\tau(t) = f(t-\tau)$. Then

$$\hat{f}_\tau(n) = \hat{f}(n) \cdot e^{-in\tau}$$

(e) $$|\hat{f}(n)| \leq \frac{1}{2\pi}\int |f(t)|\,dt = \|f\|_{L^1}.$$

Cor. Assume $f_j \in L^1(\mathbb{T})$. $j=0,1,\dots$ and $\|f_j - f_0\|_{L^1} \to 0$.

Then $\hat{f}(n) \to \hat{f}_0(n)$ uniformly.

Thm. Let $f \in L^1(\mathbb{T})$, assume $\hat{f}(0)=0$, and define

$$F(t) = \int_0^t f(\tau)\,d\tau.$$

Then F is continuous, 2π-periodic, and

$$\hat{F}(n) = \frac{1}{in}\hat{f}(n).$$

Thm. For $f, g \in L^1(\mathbb{T})$, define the convolution of f and g to be

$$(f*g)(t) = \frac{1}{2\pi}\int f(t-\tau)\, g(\tau)\, d\tau .$$

Then
$$(f*g) \in L^1(\mathbb{T})$$
and
$$\|f*g\|_{L^1} \le \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

Moreover,
$$\widehat{(f*g)}(n) = \hat{f}(n) \cdot \hat{g}(n).$$

Thm. The convolution (as defined in the thm above) is commutative, associative and distributive (w.r.t the addition).

Proof: Mostly routine. Leave as an exercise. $\square$

Lemma Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for some $n \in \mathbb{Z}$. Then
$$(\varphi * f)(t) = \hat{f}(n)\, e^{int}.$$

Proof: Routine. $\square$.

DEFN A summable kernel is a sequence $\{k_n\}$ of continuous 2π-periodic functions s.t

1) $\frac{1}{2\pi}\int k_n(t)\,dt = 1$.

2) $\frac{1}{2\pi}\int |k_n(t)|\,dt < \infty$.

3) $\forall\ 0 < \delta < \pi$,

$$\lim_{n\to\infty}\int_{\delta}^{2\pi-\delta} |k_n(t)|\,dt = 0 .$$

If k_n are all positive, then we say $\{k_n\}$ is a positive summable kernel.

The following lemma is easy to prove, but it is essential for the Fourier transformations.

Lemma. Let B be a Banach space, φ a continuous B-valued function on $\mathbb{T}$, and $\{k_n\}$ a summable kernel. Then

$$\lim_{n\to\infty}\frac{1}{2\pi}\int k_n(t)\varphi(t)\,dt = \varphi(0) .$$

Following this Lemma, we ~~have~~ have this important theorem, which indicates that we can do "approximations" in Banach spaces.

<u>Notation</u>: For f ~~$\in C(\mathbb{T})$~~, let

$$f_\tau(t) = f(t-\tau).$$

<u>Theorem</u>. Let $f \in$ ~~$C(\mathbb{T})$~~ $L^1(\mathbb{T})$ and $\{K_n\}$ be a summable kernel. Then

$$f = \lim_{n\to\infty} \frac{1}{2\pi}\int K_n(\tau)\, f_\tau \, d\tau$$

in the <u>$L^1(\mathbb{T})$ norm</u>.

<u>Proof</u>: Verifications only. $\square$.

<u>Remark</u>: This theorem is about convergence in $L^1(\mathbb{T})$ norm.

<u>Lemma</u>: Let K be a continuous function on $\mathbb{T}$ and $f \in L^1(\mathbb{T})$. Then

$$\frac{1}{2\pi}\int K(\tau)\, f_\tau \, d\tau = K * f.$$

<u>Remark</u>: We are talking about B-valued L^1 functions defined on $\mathbb{T}$.

Now, we show that approximation in <u>a "broader" Banach space</u> is doable by showing the following theorem. [<u>Remark</u>: The Theorems and Lemmas so far already ensures Fourier transformations / approximations on $C(\mathbb{T})$. <u>The following theorem is to ensure such approximations exist even for</u> $L^p(\mathbb{T})$, etc]

Theorem. Let B be a linear subspace of $L^1(\mathbb{T})$, having a norm $\|\cdot\|_B \geq \|\cdot\|_{L^1}$ under which it is a Banach space, and having the following properties:

1) If $f \in B$ and $\tau \in \mathbb{T}$, then

$$f_\tau \in B \text{ and } \|f_\tau\|_B = \|f\|_B.$$

where $\quad f_\tau(t) := f(t-\tau)$.

2) For all $f \in B$, $\tau, \tau_0 \in \mathbb{T}$,

$$\lim_{\tau \to \tau_0} \|f_\tau - f_{\tau_0}\| = 0.$$

Let $f \in B$ and let $\{K_n\}$ be a summable kernel. Then

$$\|K_n * f - f\| \to 0 \quad \text{as} \quad n \to \infty.$$

Remark: The proof of this Theorem is omitted. The main thing is to use the fact that $\|\cdot\|_B \geq \|\cdot\|_{L^1}$ and apply previous results. The proof itself is <u>not</u> difficult.

Applications

E.g. (of summable kernel)

One of the most famous summable kernel is the Fejér kernel defined as

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}.$$

Plugging in the Fejér kernel to the results above, we get that for any $f \in$ ~~C[0,1]~~ $C(\mathbb{T})$.

$$f = \lim_{n \to \infty} K_n * f .$$

This is exactly "approximating f in the $\|\cdot\|_{C(\mathbb{T})}$ norm by relatively simple functions $K_n * f$, which are trigonometric polynomials"